Talk 6: Representability of the Picard Functor - Part I

Orally: what we want to do / going to do in this talk + strategy.

§ Goal of this talk / Definitions

Def (rel. Picard functor) Let $f: X \to k$ proj, sm, irred. curve/k - alg closed.
Fix some $\sigma \in X(k)$. Then we define:

$$\mathrm{Pic}_{X/k,\sigma} : (\mathrm{Sch}/k)^{op} \to (\mathrm{Sets})$$

$$T \mapsto \ker(\sigma_T^*: \mathrm{Pic}(X_T) \to \mathrm{Pic}(T))$$

where $\sigma_T = (\sigma \times id_T): T \to X_T$.

Goal: $\mathrm{Pic}_{X/k,\sigma}$ is repres. by a scheme!

↳ Strategy: • define an open subfunctor $F \subseteq \mathrm{Pic}_{X/k,\sigma}$ which is representable (this talk!)

next talk: • ~~[illegible]~~ use group str. of $\mathrm{Pic}_{X/k,\sigma}$ to „translate F around" to produce a cov. ~~[illegible]~~

⇒ then apply representability criterion that was used before for repres. of Hilbert-functor.

$\left\{ \begin{array}{l} \text{a) Zariski sheaf} \\ \text{b)} \ldots \end{array} \right.$

Rmk 1: Since σ_T^* is section of $\mathrm{Pic}(T) \xrightarrow{\pi_T^*} \mathrm{Pic}(X_T)$

$$\Rightarrow \mathrm{Pic}(X_T) \simeq \mathrm{Pic}(T) \oplus \underbrace{\ker(\sigma_T^*)}_{= \mathrm{Pic}_{X/k,\sigma}(T)}$$

could also define relative Picard functor $\mathrm{Pic}_{X/k}$ via $T \mapsto \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$
this is isomorphic to $\mathrm{Pic}_{X/k,\sigma}$.
(later convenient to have the other defi on hand too).

Rmk 2: ~~also~~ can't take absolute Picard-functor $T \mapsto \mathrm{Pic}(X_T)$
bc. this is in general not even a Zariski-sheaf.

bc: Given a ~~line~~ line bundle $L \in \mathrm{Pic}(T)$, s.th. $f_T^* L$ is non-trivial
and a cover $\{U_i\}_{i\in I}$ of T that trivializes L
then $f_T^* L$ pulled back to X_{U_i} is trivial

$$\Rightarrow \text{the map} \quad \mathrm{Pic}(X_T) \to \prod_i \mathrm{Pic}(X_{U_i}) \quad \text{is not}$$
$$\mathcal{L} \mapsto \mathcal{L}|_{X_{U_i}}$$

injective. $\Rightarrow$ not Zariski sheaf.

§ the open subfunctor $F \subseteq \mathrm{Pic}_{X/k,\sigma}$:

$\rightsquigarrow$ first intuitive, then defi.

3

<u>Motivation for Definition of (open) subfunctor</u>: X sm, proj curve, genus g / k - alg closed

Consider the Abel-Jacobi-map:

line bundles on X of degree g.

$$AJ: \{\text{effective } CD \text{ on } X \text{ deg } g\} \longrightarrow \operatorname{Pic}^g(X)$$

$$\begin{array}{ccc} \| & D \mapsto \mathcal{O}_X(D) & \text{in} \\ \operatorname{Hilb}^g_{X/k}(k) & & \operatorname{Pic}_{X/k,\sigma}(k) \end{array}$$

this map is surjective (Talk 1, Cor 5.5, mainly RR.)

We also know:

finite dim. k-vsp

$$\{\text{eff. cart divisors on } X\} \overset{1:1}{\leftrightarrow} \{(\mathcal{L},s),\ s \in H^0(X,\mathcal{L})\setminus\{0\}\}/\cong$$

$$D \longmapsto (\mathcal{O}_X(D), 1_D) \qquad 1_D: \mathcal{I}_D \hookrightarrow \mathcal{O}_X$$

$$V(s) \longleftarrow (\mathcal{L},s)$$

Fixing $\mathcal{L}$:

finite dim k-vsp.

$$\{\text{eff. Cart. divisors } D \text{ with } \mathcal{O}(D) \cong \mathcal{L}\} \cong H^0(X,\mathcal{L})\setminus\{0\}/k^\times$$

therefore: for AJ to be an iso would need $\dim_k H^0(X,\mathcal{L}) = 1$

(in that case by (weak) RR (talk 1): $\underbrace{\dim(H^0(X,\mathcal{L}))}_{1} - \dim(H^1(X,\mathcal{L})) = \overbrace{\deg(\mathcal{L})}^{=g} + 1 - g$

$\Rightarrow \dim(H^1(X,\mathcal{L})) = 0$.)

Def + lemma between

Lemma 4 [0B9VJ] Let $T \in Sch/k$. We define a subset of $\operatorname{Pic}_{X/k,\sigma}$ by: $[f: X \to \operatorname{Spec}(k)]$ St. morph.

$T = \operatorname{Spec}(R)$

$$F(T) := \left\{ \mathcal{L} \in \operatorname{Pic}_{X/k,\sigma}(T) \;\middle|\; H^i(X_T,\mathcal{L}) = \begin{cases} 0 & \text{if } i \neq 0 \\ M. & \text{if } i = 0 \end{cases} \right\}$$

finite proj R-module of rank 1

Then $F \subseteq \operatorname{Pic}_{X/k,\sigma}$ is an open subfunctor.

Reduce to T/k affine

~~Why can we reduce to T/k affine~~? F Zariski sheaf + gluing.

~~Zariski~~

Explain the difference to Stacks proj + following lemma!

in stacks proj: for the proof that $F \subseteq \mathrm{Pic}_{X/S,\tau}$ is open subfunctor, they use derived cat / fun. bas.
→ not nec. in our situation, therefore more explicit lemma:

Lemma [Blackbox]: $f: X \to S = \mathrm{Spec}(R)$ proper, smooth of rel. dim 1, $\mathcal{L} \in \mathrm{Pic}(X)$

Then there exists a complex $K_0 \to K_1$ ($\simeq Rf_*(\mathcal{L})$ (in derived cat)) (more general: fin. pres. $\mathcal{O}_X$-mod) of finite proj. R-modules, s.th.:

$\forall$ R-algebra R', $X_{R'} = X \times_R R'$, $\mathcal{L}'$ pullback of $\mathcal{L}$ to $X_{R'}$, we have isos:

($\simeq g^* Rf_*(\mathcal{L})$, $g: \mathrm{Spec}(R') \to \mathrm{Spec}(R)$)

$$H^i(R' \otimes_R K_0 \to R' \otimes_R K_1) \simeq H^i(X_{R'}, \mathcal{L}')$$

$$\left(\simeq H^0(\mathrm{Spec}(R'), R^i f_{\mathrm{Spec}R',*}(\mathcal{L}'))\right)$$

proof (lemma 2)

every ~~corr.~~ morph of Pic restricts to morp of F.

1) subfunctor: F is stable under BC; (essentially follows from Blackbox-lemma)

Let $g: T' \to T$ be a morph of schemes, $\mathcal{L} \in F(T)$. Ts: $g^*\mathcal{L} \in F(T')$.
$T' = \mathrm{Spec}(R')$, $T = \mathrm{Spec}(R)$

proof: By BB we know that $\exists$ complex $K_0 \to K_1$ s.th. $H^i(K_0 \to K_1) \simeq H^i(X_T, \mathcal{L})$

$\mathcal{L} \in F(T)$

$\Rightarrow H^i(K_0 \to K_1) \simeq \begin{cases} 0 & i \neq 0 \\ M & i = 0 \end{cases}$ (fin, proj R-mod rk 1.)

from BB applied to $f_T: X_T \to T$ (sm. proper rel. dim 1), $\mathcal{L} \in \mathrm{Pic}(X_T)$ $\exists$ $(K_0 \to K_1)$ complex sth.

$\Rightarrow R \xrightarrow{id} R$: $H^i(K_0 \to K_1) \simeq H^i(X_T, \mathcal{L})$

$\Rightarrow R \xrightarrow{g} R'$: $H^i(R' \otimes K_0 \to R' \otimes K_1) \simeq H^i(X_{T'}, g^*\mathcal{L})$

⇒ have split ses: $0 \to \ker(\psi) \to K_0 \xrightarrow{\psi} K_1 \to 0$, with $\ker(\psi) = M$

any R'/R

$0 \to R' \otimes_R \ker(\psi) \to R' \otimes K_0 \xrightarrow{\psi \otimes id_{R'}} R' \otimes K_1 \to 0$
split, exact.

(esp: $R' \otimes_R \ker(\psi) \simeq \ker(\psi \otimes id_{R'})$)

⇓

$$H^i(X_{T'}, g^*\mathcal{L}) \simeq H^i(R' \otimes (K_0 \to K_1)) = \begin{cases} 0 & i \neq 0 \\ \underbrace{R' \otimes_R M}_{\text{fin. proj. } R'\text{-mod rank 1}} & i = 0 \end{cases}$$

$\Rightarrow g^*\mathcal{L} \in F(T')$ #

2) $F \subseteq \mathrm{Pic}_{X/k,\sigma}$ open subfunctor

Let $T = \mathrm{Spec}(R)$, $h \in \mathrm{Pic}_{X/k}(T)$ and $(K_0 \to K_1)$ complex from Black Box Lemma (complex of finite, proj R-mod)

claim: $\exists U \subseteq T$ open s.th.

$$\begin{tikzcd} \text{any } T' \arrow[r, "g"] \arrow[dr, dashed] & T & \text{factors} \\ & U \arrow[u, "\subseteq_o"] & \end{tikzcd} \Leftrightarrow \underbrace{H^i(X_{R'}, g^*h)}_{= \mathrm{Spec}(R')} \overset{H^i(R'\otimes(K_0\to K_1))}{\overset{\wr\!\!\mid}{\cong}} \begin{cases} 0 & i \neq 0 \\ \text{proj. } R'\text{-mod rk } 1 & i = 0 \end{cases}$$

$\Leftrightarrow g^*h \in F(T')$

proof:

have an exact sequence $K_0 \xrightarrow{\varphi} K_1 \to \underset{\mathrm{Coker}(\varphi)}{\overset{\shortparallel}{Q}} \to 0$. of R-mod.

then $\mathrm{supp}(Q) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid Q_{\mathfrak{p}} \neq 0\}$ is closed in T. $(*)$

$\Rightarrow$ take $U = (\mathrm{Supp}(Q))^c \subseteq_o T$ and check above desired cond. ✓

$(*)$ locally φ looks like: $R^m \xrightarrow{(a_{ij}) = A} R^n$

and $x \notin \mathrm{Supp}\, Q \Leftrightarrow A(x) \in M(k(x))$ has max. rank

$\Leftrightarrow \exists\ n \times n$ minor with $\det(x) \neq 0$

↑ this is an open cond. #

□

Def: Let k be field, X sm, proj irred. curve$/k$. The genus of X is defined as: $g = \dim_k H^1(X, \mathcal{O}_X)$.

<u>Example (genus of $\mathbb{P}^1$)</u> Consider the standard cover $X = \mathbb{P}^1 = \underset{\mathbb{A}^1}{\underset{\cong}{U_1}} \cup \overset{\mathbb{A}^1}{\overset{\cong}{U_2}}$

Have a long exact sequence (Mayer-Vietoris) (for general ringed space X with $X = \underbrace{U \cup V}_{\text{open}}$, $\mathcal{E} \in \mathcal{O}_X\text{-Mod}$)

$$0 \to H^0(X, \mathcal{O}_X) \to \overset{k[T]}{\overset{\cong}{H^0(U_1, \mathcal{O}_{U_1})}} \oplus \overset{k[T^{-1}]}{\overset{\cong}{H^0(U_2, \mathcal{O}_{U_2})}} \xrightarrow{+} \overset{k[T^{\pm 1}]}{\overset{\cong}{H^0(U_1 \cap U_2, \mathcal{O}_{U_1 \cap U_2})}} \to \dots$$

$$\dots \xrightarrow{\psi} H^1(X, \mathcal{O}_X) \to 0$$

Talk 1: Cohomology vanishes for affine X and $\mathcal{E} \in \mathrm{QCoh}(X)$

Since $k[T^{\pm 1}] = \mathrm{im}(+) = \ker(\psi)$

$$\Rightarrow \psi = 0 \Rightarrow H^1(X, \mathcal{O}_X) = 0$$

$$\Rightarrow g(\mathbb{P}^1) = H^1(X, \mathcal{O}_X) = 0 \quad \square$$

§3. Representability of $\mathcal{F}$

Lemma [0B9X] k field. X sm proj. genus g/k, $\sigma \in X(k)$. Then $\mathcal{F}$ is repres. by an open subscheme of $\underline{\mathrm{Hilb}}^g_{X/k}$.

Recall: $\mathrm{Hilb}^g_{X/k} : \mathrm{Sch}/k \longrightarrow \mathrm{Sets}$, $T \mapsto \{ U \subseteq T \times_k X$ clsd subsch $\cdot\ U \to T$ fin. loc. free deg $g \}$

talk 3 $\Rightarrow$ this is repres. by a scheme! call it H.

H - our situation, esp. $X \to k$ rel dim 1 (Talk 4) $\{$ rel. eff. CD D on $T \times_k X$ sth. $D \to T$ is fin loc. free deg $g \}$

proof. strategy: want to obtain $W \underset{o}{\subseteq} H$ (by openness of $\mathcal{F} \subseteq \mathrm{Pic}$) and specific maps sth. we have a comm diag:

$$\begin{tikzcd}
② W \arrow[r, dashed, "\psi ③"] \arrow[d] & \mathcal{F} \arrow[d, "\mathrm{incl.}"] \\
H \arrow[r, "①"'] & \mathrm{Pic}_{X/k,\sigma}
\end{tikzcd}$$

① define this map (scheme-version of Abel-Jakobi) $D \mapsto \mathcal{O}_X(D)$

② get diagram with $W \underset{o}{\subseteq} H$

③ ψ is an iso! $\Rightarrow$ repres. of $\mathcal{F}$.

now details:

① Consider the universal divisor $D_{uni} \subseteq H \times_k X$ $\rightsquigarrow$ assoc. inv. sheaf (on $H \times_k X$): $\mathcal{O}(D_{uni})$

we „adjust" $\mathcal{O}(D_{uni})$ to obtain an elem in $\mathrm{Pic}_{X/k,\sigma}$:

$$h_H := \mathcal{O}(D_{uni}) \otimes_{\mathcal{O}_{H \times X}} f_H^* \sigma_H^* \mathcal{O}(D_{uni})^\vee \in \mathrm{Pic}_{X/k,\sigma}(H).$$

mainly bc. pullback commutes with tensor..

By Yoneda $\left(\mathrm{Nat}(H, \mathrm{Pic}_{X/k,\sigma}) \xrightarrow{\sim} \mathrm{Pic}_{X/k,\sigma}(H) \right)$

$$F \mapsto F(H)(\mathrm{id}_H)$$

$$h_H \leftrightarrow \text{nat trafo } H \xrightarrow{h_H} \mathrm{Pic}_{X/k,\sigma}.$$

precisely given on points: $T \in \mathrm{Sch}/k$:

$$H(T) \longrightarrow \mathrm{Pic}_{X/k,\sigma}(T)$$
$$[T \xrightarrow{f} H] \mapsto (f \times \mathrm{id}_X)^* h_H.$$

#

Let $T \in Sch/k$. We know $F \subseteq Pic_{X/k,\sigma}$ open subfunctor, therefore for every pair (H, h_H) there exists an open $W \overset{\subseteq}{\circ} H$ sth:

$$\text{any } \begin{tikzcd} Y \arrow[r] \arrow[dr, dotted] & H \\ & W \arrow[u, phantom, "\subseteq"] \end{tikzcd} \text{ factors} \iff h_H|_{Y\times X} \in F(Y)$$

Setting $Y = W \Rightarrow h_H|_{W\times X} \in F(W)$

$\updownarrow$ Yoneda

nat trafo $W \to F$

$\xrightarrow[\text{Yoneda}]{\text{naturality}}$

$$\begin{tikzcd} W(T) \arrow[r, "\Psi"] \arrow[d] \arrow[dr, phantom, "//"] & F(T) \arrow[d, hook, "\text{incl.}"] \\ H(T) \arrow[r, "h_H"] & Pic_{X/k,\sigma}(T) \end{tikzcd}$$

③ Ψ is iso:

first: given $h \in F(T)$, N inv. O_T-mod sth. $f_{T*} h \cong N$, have adj. map

$$f_T^* N \to h \overset{\text{corresp.}}{\iff} \text{global section } s \in h \otimes (f_T^* N)^\vee .$$

claim: zero scheme of s - rel. eff. CD on X_T/T, fin. loc. free of deg g/T (proof later)

Since H classifies rel. effect. CD $\Rightarrow D \overset{\text{corresp}}{\iff} m: T \to H$ with $(m \times id)^* O(D_{uni}) \cong O_{X_T}(D)$.

claim: m factors over $W \overset{\subseteq}{\circ} H$ and $\Psi(m) = h$ ($\to$ surjectivity of Ψ!) first diagram!

- by defi $O(D) \cong h \otimes (f_{T*} h)^\vee$ and $[h] = [h \otimes (f_{T*} h)^\vee]$ in $Pic_{X/k}(T) = Pic(X_T)/Pic(T)$
- $[O(D)] = [h|_{T\times H}]$ in $Pic_{X/k}(T)$ by defi.

$\Rightarrow$ in pk. $h_H|_{T\times H} \in F(T) \underset{T \subseteq Pic_{X/k,\sigma}}{\overset{\text{openness}}{\iff}}$ $\begin{tikzcd} T \arrow[r, "m"] \arrow[dr, dotted] & H \\ & W \arrow[u, phantom, "\subseteq"] \end{tikzcd}$ factors.

therefore we get:

$$\begin{tikzcd}
m \arrow[d] & W(T) \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "///"] & \mathcal{F}(T) \arrow[d, hook] & h \arrow[d, mapsto] \\
m \in H(T) \arrow[r, "h_H"] \arrow[d, phantom, "\wr\|"] & \mathrm{Pic}_{X/k,=}(T) \arrow[d, phantom, "\|\wr \; \mathrm{I}"] & & \mathcal{L} \\
D \in \mathrm{Hilb}^g(T) \arrow[r, "AJ"] \arrow[u, leftrightarrow] & \mathrm{Pic}_{X/k}(T) & &
\end{tikzcd}$$

$m \longmapsto h|_{T\times X}$

$D \longmapsto [\mathcal{O}(D)]$

$\Rightarrow \psi(m) = h$ and ψ surj ✓

ψ <u>injective:</u> Assume there is another rel. eff. CD D' with $\mathcal{O}(D') \cong h$

Let $s' \in H^0(X_T, \mathcal{O}(D'))$ be the regular section def. D', have:

$$s' \in H^0(X_T, \mathcal{O}(D')) \cong H^0(X_T, \overset{\cong h}{\mathcal{O}(D)}) \ni s$$

restriction to fibre

$\left(\begin{array}{l} H^0(X_T, \mathcal{O}(D')) \text{ comm} \\ \text{with base change} \\ \text{b.c. } \mathcal{O}(D') \in \mathcal{F}(T) \end{array}\right)$

$$s'| \in H^0(\underbrace{(X_T)_t, \mathcal{O}(D')|}_{\cong k}) \cong \underbrace{H^0((X_T)_t, \mathcal{O}(D)|)}_{\cong k} \ni s|$$

$$\Rightarrow s'| = \underset{k^*}{\overset{\in}{\lambda}} \cdot s| \quad \Rightarrow \text{fibrewise same effective rel CD.} \quad \Rightarrow D' = D$$

<u>left to s:</u> $D = V(s)$, $s \in h \otimes (f_T^* N)^\vee$ is rel. eff. CD of deg g. (= genus of X)

by [062Y] it suffices to show that the fibres of D over T are effective CD. (of degree g over k.)

Since $\overset{\cong \mathcal{O}(D)}{h} \in \mathcal{F}(T)$ $\Rightarrow$ taking cohomology commutes with b.c.

Therefore, we reduce to $\mathrm{Spec}(k)$: h invertible sheaf on X with $H^0(X, \mathcal{L}) = k$, $H^1(X, h) = 0$

$$\Rightarrow \deg(\mathcal{L}) = \chi(X, h) - \chi(X, \mathcal{O}_X) = 1 - (1-g) = g$$

weak RR (Talk 1) $\Rightarrow \deg(h) = \deg(D) \Rightarrow D$ is eff. CD (of deg g over k) #