

Talk 6: Representability of the Picard Functor - Part I

Orally: What we want to do / going to do in this talk + stuff.

§ Goal of this talk / Definitions

Def (rel. Picard functor) Let $f: X \rightarrow k$ proj, sm, irred. connex - alg closed.

Fix some $\sigma \in X(k)$. Then we define:

$$Pic_{X/k, \sigma} := (S_k/k)^{op} \rightarrow (Sets)$$

$$T \mapsto h_T(\sigma_T^*: Pic(X_T) \rightarrow Pic(T))$$

where $\sigma_T = (\sigma \times id_T): T \rightarrow X_T$.

Goal: $Pic_{X/k, \sigma}$ is repres. by a scheme!

↳ strategy: • define an open subfunctor $F \subseteq Pic_{X/k, \sigma}$ which is representable (this talk!)

next talk: • ~~use~~ use group str. of $Pic_{X/k, \sigma}$ to "translate F around" to produce a cong. ~~to~~

=> then apply representability criteria that was used before for repres. of H¹-funct.

- { a) Zariski sheaf
- { b) ..

Remark 1: Since σ_T^* is section of $\text{Pic}(T) \xrightarrow{\pi_T^*} \text{Pic}(X_T)$

$$\Rightarrow \text{Pic}(X_T) \cong \text{Pic}(T) \oplus \underbrace{\text{Ker}(\sigma_T^*)}_{= \text{Pic}_{X/T}(T)}$$

could also define relative Picard sheaf $\text{Pic}_{X/T}$ via $T \rightarrow \text{Pic}(X_T) / \text{Pic}(T)$
this is isomorphic to $\text{Pic}_{X/T}$.

(later convenient to have the other def on hand too).

Remark 2: ~~also~~ can't take absolute Picard-sheaf $T \rightarrow \text{Pic}(X_T)$

bc this is in general not even a sheaf.

5c: Given a ~~line~~ line bundle $L \in \text{Pic}(T)$, sth. $\pi_T^* L$ is non-trivial
and a cover $\{U_i\}_{i \in I}$ of T that trivializes L
then $\pi_T^* L$ pulled back to X_{U_i} is trivial

\Rightarrow the map $\text{Pic}(X_T) \rightarrow \prod_i \text{Pic}(X_{U_i})$ is not

$$\mathcal{L} \mapsto \mathcal{L}|_{X_{U_i}}$$

injective.

\Rightarrow not sheaf.

§ The open subsheaf $F \subseteq \text{Pic}_{X/T}$:

\hookrightarrow first inclusion, then def.

Motivation for Definition of (open) subfunctor: X sm, proj curve, genus g / k-aly closed

Consider the Abel-Jacob map:

line bundles on X
of degree g .

$$\begin{aligned} A_J: \{ \text{Effective } (D \text{ on } X \text{ of } g) \} &\rightarrow \text{Pic}^g(X) \\ \parallel \quad D &\mapsto \mathcal{O}_X(D) \text{ in} \\ \text{Hilb}_{X/k}^g(h) &\quad \text{Pic}_{X/k, \sigma}(h) \end{aligned}$$

this map is surjective (Taak 1, cor 5.5, mainly RR.)

We also know:

$$\begin{aligned} \{ \text{eff. cart. divisors on } X \} &\xrightarrow{\cong} \{ (\mathcal{L}, s) \mid s \in H^0(X, \mathcal{L}) \setminus \{0\} \} / \sim \\ D &\mapsto (\mathcal{O}_X(D), 1_D) \quad 1_D \cdot \lambda_D \mapsto \mathcal{O}_X \\ V(s) &\leftarrow (h, s) \end{aligned}$$

finite dim. k-vsp.

Fixing \mathcal{L} :

$$\{ \text{eff. cart. divisors } D \text{ with } \mathcal{O}(D) \cong \mathcal{L} \} \cong H^0(X, \mathcal{L}) \setminus \{0\} / k^\times$$

finite dim. k-vsp.

therefore: for A_J to be an iso would need $\dim_k H^0(X, \mathcal{L}) = 1$

(heavily)

$$\left(\begin{array}{l} \text{in that case by RR: } \dim(H^0(X, \mathcal{L})) - \dim(H^1(X, \mathcal{L})) = \deg(\mathcal{L}) + 1 - g \\ \text{(talk 1)} \quad \quad \quad \uparrow \quad \quad \quad \Rightarrow \dim(H^1(X, \mathcal{L})) = 0. \end{array} \right)$$

Def + lemma between

Lemma 2 Let $T \in \text{Spec } k$. We define a subset of $\text{Pic}_{X/k, \sigma}$ by: $T = \text{Spec}(R)$

$[T: X \rightarrow \text{Spec}(k)]$
St: morph.

$$F(T) := \{ \mathcal{L} \in \text{Pic}_{X/k, \sigma}(T) \mid H^i(X_T, \mathcal{L}) = \begin{cases} 0 & \text{if } i > 0 \\ M & \text{if } i = 0 \end{cases} \}$$

then $F \subseteq \text{Pic}_{X/k, \sigma}$ is an open subfunctor.

finite proj
R-module of
rank 1

Relative to T/k affine

Why can we reduce to this affine? F Zariski sheaf + geny.

Explain the difference to } in stacks proj: for the proof that $F \in \text{Pic}_{X/R} \Rightarrow$ open
 stacks proj + Bllow lemma! } subfunctor, they use derived cat / complexes.
 \rightarrow not nec. in our situation, therefore use explicit lemma:

Lemma [Blackbox]: $f: X \rightarrow S = \text{Spec}(R)$ proper, smooth of rel. dim 1, $L \in \text{Pic}(X)$
 Then there exists a complex $K_0 \rightarrow K_1$ of finite proj. R -modules, s.t.:

$\forall R$ -algebra R' , $X_{R'} = X \times_R R'$, L' pullback of L to $X_{R'}$, we have isos:

$$H^i(R' \otimes_R K_0 \rightarrow R' \otimes_R K_1) \cong H^i(X_{R'}, L')$$

$$\left(\cong H^0(\text{Spec}(R'), R' \otimes_{\text{Spec}(R')} (L')) \right)$$

proof (lemma 2) $\xrightarrow{\text{anal}}$ wop of Pic restricts to wop of F .

1) subfunctor: F is stable under BC; (essentially follows from Blackbox-lemma)

Let $g: T' \rightarrow T$ be a morph of schemes, $L \in F(T)$. $T_S: g^*L \in F(T')$.
 $\text{Spec}(R') \quad \text{Spec}(R)$

proof By BB we know that f complex $K_0 \rightarrow K_1$ s.t. $H^i(K_0 \rightarrow K_1) \cong H^i(X_T, L)$

\Rightarrow have split ses: $0 \rightarrow \text{ker}(\psi) \rightarrow K_0 \xrightarrow{\psi} K_1 \rightarrow 0$

any R'/R \downarrow $0 \rightarrow R' \otimes_R \text{ker}(\psi) \rightarrow R' \otimes_R K_0 \xrightarrow{\psi \otimes \text{id}_{R'}} R' \otimes_R K_1 \rightarrow 0$
 split, exact.

(esp: $R' \otimes_R \text{ker}(\psi) \cong \text{ker}(\psi \otimes \text{id}_{R'})$)

\Downarrow
 $H^i(X_{T'}, g^*L) \cong H^i(R' \otimes (K_0 \rightarrow K_1)) = \begin{cases} 0 & i \neq 0 \\ R' \otimes H & i=0 \end{cases}$
 $\Rightarrow R' \otimes R: H^i(K_0 \rightarrow K_1) \cong H^i(X_T, L)$
 $\Rightarrow R' \otimes R: H^i(R' \otimes K_0 \rightarrow R' \otimes K_1) \cong H^i(X_{T'}, g^*L)$
 $\Rightarrow g^*L \in F(T') \quad \#$

$\parallel S \quad L \in F(T)$
 $\begin{cases} 0 & i \neq 0 \\ H & i=0 \end{cases}$
 $\leftarrow \text{fin. proj. } R\text{-mod rk 1.}$
 $\leftarrow \text{sin. proper rel. dim 1.}$

how BB applied to $f: X_T \rightarrow T$
 $L \in \text{Pic}(X_T) \ni (K_0 \rightarrow K_1)$ complex s.t.
 $\Rightarrow R' \otimes R: H^i(K_0 \rightarrow K_1) \cong H^i(X_T, L)$

2) $\mathbb{F} \subseteq \text{Pic}_{X/Y, r}$ open substack

Let $T = \text{Spec}(R)$, $h \in \text{Pic}_{X/Y}(T)$ and $(U_0 \rightarrow U_1)$ complex of locally free R -mod
Lemma (complex of finite, proj R -mod)

claim: $\exists U \subseteq T$ open s.t.

$$\begin{array}{c} \text{any } T' \xrightarrow{g} T \\ \swarrow \quad \searrow \\ \quad \quad U \xrightarrow{k_0} \end{array} \text{ factors } \Leftrightarrow H^i(X_{R'}, g^* \mathcal{L}) \cong \begin{cases} 0 & i \neq 0 \\ \text{proj. } R' \text{-mod} & i = 0 \\ \text{rk } 1 \end{cases}$$

$\Leftrightarrow g^* \mathcal{L} \in \mathbb{F}(T')$

proof:

have an exact sequence $k_0 \rightarrow k_1 \rightarrow Q \rightarrow 0$ of R -mod.
" $\text{Coloc}(+)$

then $\text{supp}(Q) = \{ \mathfrak{p} \in \text{Spec}(R) \mid Q_{\mathfrak{p}} \neq 0 \}$ is closed in T . (*)

\Rightarrow take $U = (\text{supp}(Q))^c \subseteq T$ and check above desired cond. \checkmark

(*) locally ψ looks like: $R^m \xrightarrow{(a_i)} R^n$

and $x \notin \text{supp } Q \Leftrightarrow A(x) \in M(A(x))$ has max. rank

$\Leftrightarrow \exists n \times n$ minor with $\det(x) \neq 0$

\uparrow this is an open cond. $\#$

□

Def: Let U be hdd, X sm, proj. curve/ U . The genus of X is defined as: $g = \dim_k H^1(X, \mathcal{O}_X)$.

Example (genus of \mathbb{P}^1) Consider the standard cover $X = \mathbb{P}^1 = U_1 \cup U_2 \stackrel{\cong}{=} \mathbb{A}^1$
 \mathbb{A}^1

Have a long exact sequence (Mayer-Vietoris) (for general rigid space X with $X = U_1 \cup U_2$, $U_i \in \mathcal{O}_X$ -Mod)

$$0 \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(U_1, \mathcal{O}_{U_1}) \oplus H^0(U_2, \mathcal{O}_{U_2}) \xrightarrow{+} H^0(U_1 \cap U_2, \mathcal{O}_{U_1 \cap U_2}) \rightarrow \dots$$

$\begin{matrix} k[T] \oplus k[T^{-1}] & \xrightarrow{+} & k[T \pm 1] \\ \parallel & \parallel & \parallel \\ k[T] & \xrightarrow{+} & k[T \pm 1] \end{matrix}$

$$\dots \xrightarrow{\psi} H^1(X, \mathcal{O}_X) \rightarrow 0$$

Since $k[T \pm 1] = \text{im}(+) = \text{ker}(\psi)$

$$\Rightarrow \psi = 0 \Rightarrow H^1(X, \mathcal{O}_X) = 0$$

$$\Rightarrow g(\mathbb{P}^1) = H^1(X, \mathcal{O}_X) = 0 \quad \square$$

Take 1:

Consider ψ vanishes

for affine X and $E \in \text{Quot}(k)$

Lemma [OB9X] & Defn: X sur proj, genus $g \geq 0$, $\sigma \in X(k)$. Then F is repres by an open subscheme of $\text{Hilb}_{X/k}^g$.

Recall: $\text{Hilb}_{X/k}^g = \text{Sch}/k \rightarrow \text{Sch}$, $T \mapsto \{ U \in T \times_k X \text{ closed subscheme} \}$
 $U \rightarrow T$ lin. loc. free $\deg g$

talk 3

\Rightarrow this is repres. by a scheme!
 call it H .

H - our scheme, esp. $X \rightarrow k$ rel $\dim 1$ (talk 4)

$\left\{ \begin{array}{l} \text{rel. eff. CD } D \text{ on } T \times_k X \text{ s.t.} \\ D \rightarrow T \text{ is lin. loc. free deg } g \end{array} \right\}$

proof strategy: want to obtain $W \subseteq H$ (by openness of $F \subseteq \text{Pic}$) and specific maps s.t. we have a comm. diag:

$$\begin{array}{ccc} \textcircled{2} W & \xrightarrow{\psi \textcircled{3}} & F \\ \downarrow & & \downarrow \text{incl.} \\ H & \xrightarrow{\textcircled{1}} & \text{Pic}_{X/k, \sigma} \end{array}$$

$\textcircled{1}$ define this map (scheme-version of deul-Jakobi)

$\textcircled{2}$ get diagram with $W \subseteq H$

$D \mapsto \mathcal{O}_X(D)$

$\textcircled{3}$ ψ is an iso! \Rightarrow repres. of F .

now details:

$\textcircled{1}$ Consider the universal driver $D_{univ} \in H \times_k X \rightsquigarrow$ assoc. inv. sheaf: $\mathcal{O}(D_{univ})$ (on $H \times_k X$)

we "adjust" $\mathcal{O}(D_{univ})$ to obtain an element in $\text{Pic}_{X/k, \sigma}$:

$$h_H := \mathcal{O}(D_{univ}) \otimes_{\mathcal{O}_{H \times_k X}} \mathcal{F}_H^* \otimes_{\mathcal{O}_H} \sigma_H^* \mathcal{O}(D_{univ}) \in \text{Pic}_{X/k, \sigma}(H).$$

By Yoneda $\left(\text{Nat}(H, \text{Pic}_{X/k, \sigma}) \xrightarrow{\sim} \text{Pic}_{X/k, \sigma}(H) \right)$
 $F \mapsto F(H)(id_H)$

\uparrow mainly bc. pullback commutes with tensor.

$$h_H \Leftrightarrow \text{nat. trans } H \xrightarrow{h_H} \text{Pic}_{X/k, \sigma}$$

precisely given on points: $T \in \text{Sch}/k$:

$$H(T) \rightarrow \text{Pic}_{X/k, \sigma}(T)$$

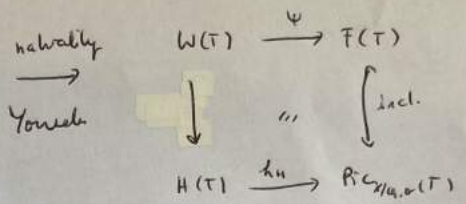
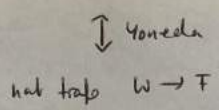
$$[T \xrightarrow{f} H] \mapsto (f \times id_X)^* h_H.$$

\neq

Let $T \in \text{Sch}/k$. We know $F \subseteq \text{Pic}_{X/k, \sigma}$ open subfunctor, therefore for every pair (H, h_H) there exists an open $W \subseteq H$ sth:

any $Y \xrightarrow{\psi} H$ factors $\Leftrightarrow \mathcal{L}_H|_{Y \times X} \in F(Y)$
 $\downarrow \psi$
 $W \hookrightarrow H$

Setting $Y = W \Rightarrow \mathcal{L}_H|_{W \times X} \in F(W)$



③ ψ is iso:

first: given $h \in F(T)$, W inv. O_T -mod sth. $f_{T \times X} h \cong W$, have adj. map

$f_T^* W \rightarrow \mathcal{L} \xleftrightarrow{\text{corresp.}} \text{global section } s \in \mathcal{L} \otimes (f_T^* W)^\vee$

claim: zero scheme of s -rel. eff. CD on X_T/T , lin. loc. free of $\text{deg } g/T$ (proof later)

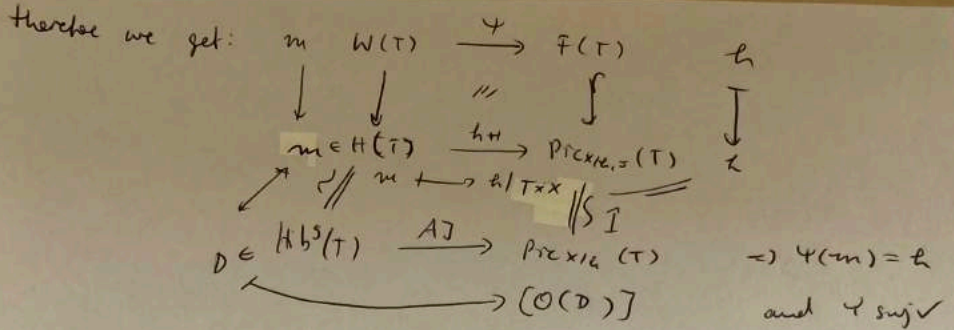
Since H classifies rel. effect. CD $\Rightarrow D \Leftrightarrow m: T \rightarrow H$ with $(m \times \text{id})^* \mathcal{O}(D_{\text{univ}}) \cong \mathcal{O}_{X_T}(D)$.

claim: m factors over $W \subseteq H$ and $\psi(m) = \mathcal{L} \Rightarrow$ surjectivity of ψ ! (not depend!)

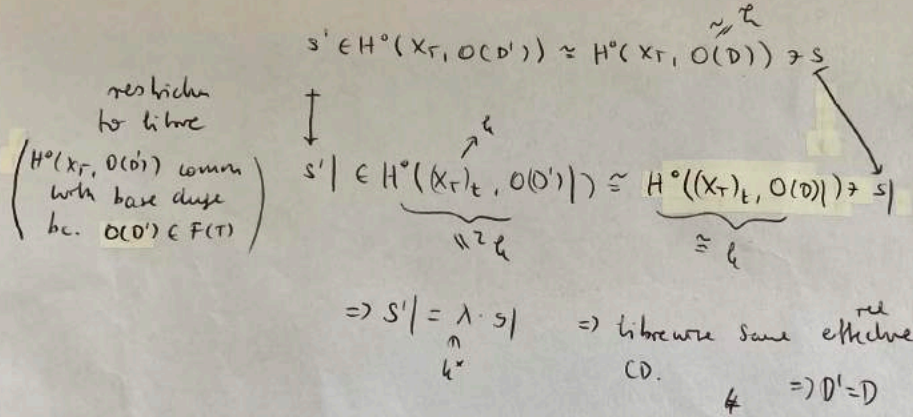
• by defn $\mathcal{O}(D) \cong \mathcal{L} \otimes (f_{T \times X} h)^\vee$ and $[h] = [\mathcal{L} \otimes (f_{T \times X} h)^\vee]$ in $\text{Pic}_{X/k, \sigma}(T) = \text{Pic}(X_T) / \text{Pic}(T)$

• $[\mathcal{O}(D)] = [\mathcal{L}|_{T \times X}]$ in $\text{Pic}_{X/k, \sigma}(T)$ by defn.

\Rightarrow in pk. $\mathcal{L}_H|_{T \times X} \in F(T) \xleftrightarrow{\text{openess}} T \xrightarrow{m} H$ factors. $\downarrow W \subseteq$



ψ injective: Assume there is another rel. eff. CD D' with $\mathcal{O}(D') \cong \mathcal{L}$.
 Let $s' \in H^0(X_T, \mathcal{O}(D'))$ be the regular section def. D' , have:



left ts: $D = V(s)$, $s \in h \otimes \mathcal{F}_T^*(\mathcal{L})$ is rel. eff. CD of deg g . (= genus of X)

by [0624] it suffices to show that the fibres of D over T are effective CD. (of degree g over k)

Since $\mathcal{L} \in \mathcal{F}(T) \Rightarrow$ taking cohomology commutes with bc.

Therefore, we reduce to $\text{Spec}(k)$: \mathcal{L} invertible sheaf on X with $H^0(X, \mathcal{L}) = h$
 $H^1(X, \mathcal{L}) = 0$

$\Rightarrow \text{deg}(\mathcal{L}) = \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X) = 1 - (1 - g) = g$

weak $\Rightarrow \text{deg}(\mathcal{L}) = \text{deg}(D) \Rightarrow D$ is eff. CD (of deg g over k)

(Talk 1)